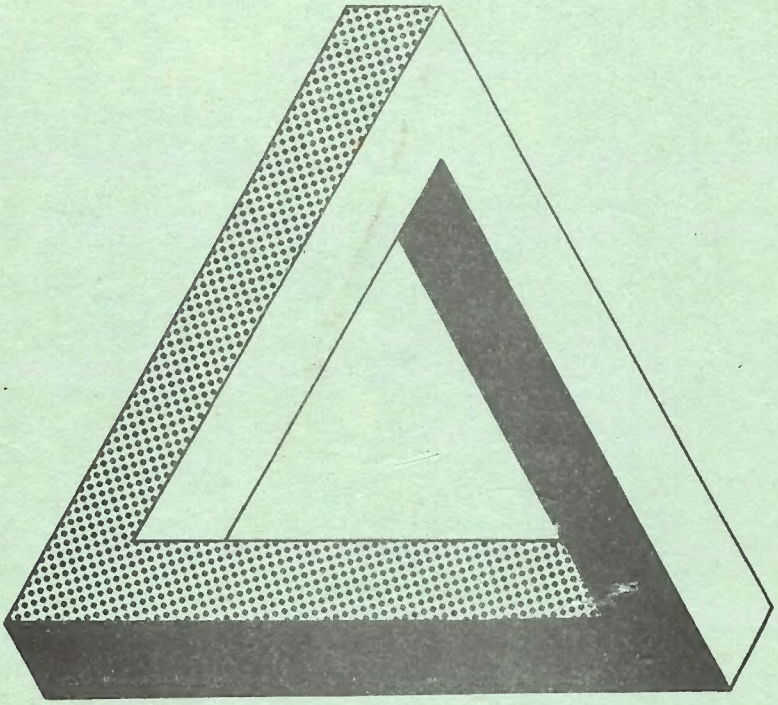
TRIGON



SCHOOL MATHEMATICS JOURNAL
OF
THE MATHEMATICAL ASSOCIATION OF
SOUTH AUSTRALIA

VOLUME 13, NO.1 MARCH 1975

EDITORIAL COMMITTEE

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J. van der Hoek

SUBSCRIPTIONS

The subscription for the three issues of TRIGON for 1975 will be 80 cents. Subscriptions should be sent to

Mr. R. Williams,
Adelaide College of Advanced Education,
Kintore Ave.,
Adelaide, 5001.

A number of back issues of the School Mathematics Journal are still available at 15 cents per copy.

READERS' CONTRIBUTIONS

The editorial committee is always delighted to hear from readers. Letters, short articles, mathematical snippets and cartoons should be sent to

The Editor, TRIGON,
Department of Mathematics,
University of Adelaide,
Adelaide, 5001.

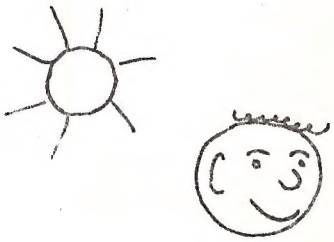
All material which is suitable for publication will be acknowledged in the Journal.

$a\left(\frac{1}{16+1+}\right.$

FROM THE EDITORS

The School Mathematics Journal has been appearing regularly now for twelve years, and we have readers in all States of Australia and even from as far away as New Zealand, South Africa and U.S.A. The non-appearance of Vol. 12, No. 4 was therefore a matter of deep regret to the Editors and to the Mathematical Association - probably to you too! It was caused by a virtual doubling of printing costs in 1974.

However, we are sure that you will enjoy the Journal in its new format and under its new title. TRIGON will appear only three times a year instead of four, but we plan to increase the total amount of material for the year. In fact, the Editors are quite excited at the prospects of introducing new features and making the most of our new freedom!



About two years ago we tried publishing solutions to the problems of Journal n in Journal $(n+1)$. This experiment failed at the time because of printing delays, but we think it might be possible this year. Hence this issue of TRIGON contains solutions to two sets of problems.

<u>Competition</u> We hope you will try the problems in each issue and send your solutions in to us. Each year we offer book prizes to the readers who submit the greatest number of correct solutions in the Junior Section and Open Section of the Problems. Juniors may count problems in the Open Section towards their total.

Book prizes have been sent to the 1974 winners:

Junior Section: David Robertson, Prince Alfred College.
Senior Section: Mark Yudkin, Marion High School.

The competition is now open for 1975, and we invite contributions.

Finally, you will notice that there has been a change in the Editorial Committee. We are sorry to lose Mr. Pender Pedler who has accepted appointment in Western Australia, but we are happy to welcome in his place Mr. Barry Dolman who is a tutor at the University of Adelaide.

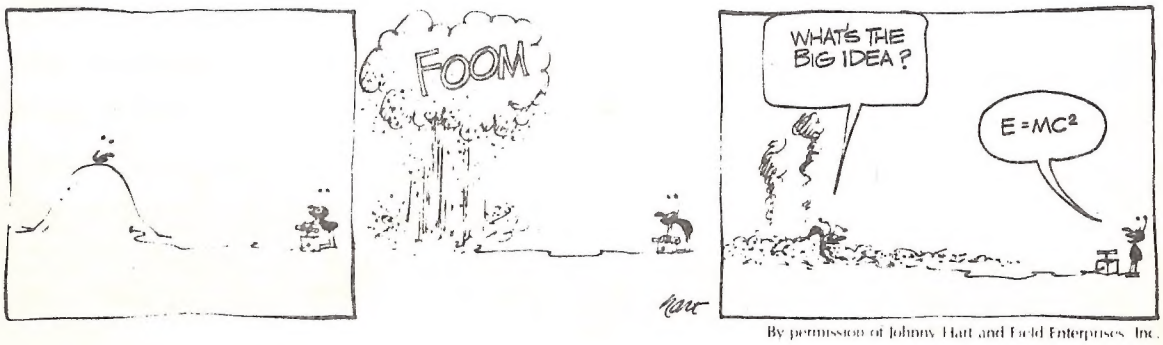
AVERAGE!

A porter at the railway station was asked by a passenger, "What's the average tip you get from people?"

"Five dollars, sir", replied the porter.

The passenger handed the porter five dollars.

"Mister," exclaimed the porter, "You're the most average person I have ever met!"



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THERE AREN'T VERY MANY PRIMES,

AND THEY ARE VERY UNEVENLY DISTRIBUTED

It may be terribly hard to convince you of the statement that is the title of this section. But have an open mind, and stare please, at the first one hundred numbers.

The primes are underlined.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Two primes are called consecutive if there are no primes between them. Thus, 7 and 11 are consecutive, while 7 and 13 are not. Also, 73 and 79 are consecutive, while 73 and 83 are not. Now here's a question. Which pairs of consecutive primes in the table are separated by at least three composites? An examination shows that there are fifteen such pairs.

7 and 11	37 and 41	67 and 71
13 and 17	43 and 47	73 and 79
19 and 23	47 and 53	79 and 83
23 and 29	53 and 59	83 and 89
31 and 37	61 and 67	89 and 97

How many are separated by at least four composites? There are naturally fewer than fifteen. Again, a glance shows they are:

23 and 29	61 and 67
31 and 37	73 and 79
47 and 53	83 and 89
53 and 59	89 and 97

making eight in all. How many are separated by at least five composites? The answer is eight.

How many pairs of consecutive primes in the table are separated by at least six composites? Only one pair of consecutive primes, 89 and 97. Notice that there are seven composites that separate 89 and 97.

Could there be consecutive primes that are separated by at least eight composites? By at least nine composites? By at least 100 composites? Is there any limit?

We start to answer these questions by looking at the following numbers: 722, 723, 724, 725, and 726. These numbers are five composites in a row. Make sure you check that they are all composite. But we already know of several examples of five composites in a row; for example, 90, 91, 92, 93, 94. So why look at another five? The reason is that a close examination of the composites 722 through 726 leads to a method that will answer the questions in the previous paragraph. The woods of mathematics are filled with cases in which looking at one example may lead to amusement or surprise, but looking at another example leads to new knowledge.

Notice that

$$722 = 2.361$$
$$723 = 3.241$$
$$724 = 4.181$$
$$725 = 5.145$$
$$726 = 6.121$$

Therefore, 2 divides 722; 3 divides 723; 4 divides 724; 5 divides 725; 6 divides 726. Other things divide those numbers also, but the divisors 2, 3, 4, 5 and 6 are suggestive of some pattern.

Look closer:

$$722 = 720 + 2,$$
$$723 = 720 + 3,$$
$$724 = 720 + 4,$$
$$725 = 720 + 5,$$
$$726 = 720 + 6,$$

and 720 = 2.3.4.5.6. In fact, recall that 720 = 6!, and there's the real clue.

If we want to find eight composites in a row, we have two choices. We can search until we find eight, or we can imitate our new-found knowledge. We noticed that

$$722 = 6! + 2,$$
$$723 = 6! + 3,$$
$$724 = 6! + 4,$$
$$725 = 6! + 5,$$
$$726 = 6! + 6,$$

gave us five composites in a row, and the factorial involved was 6. To get eight in a row, we imitate by writing

$$9! + 2 = 362\ 882$$
$$9! + 3 = 362\ 883$$
$$9! + 4 = 362\ 884$$
$$9! + 5 = 362\ 885$$
$$9! + 6 = 362\ 886$$
$$9! + 7 = 362\ 887$$
$$9! + 8 = 362\ 888$$
$$9! + 9 = 362\ 889.$$

Make sure you see the general reason why all these are composite.

We actually could have found eight composites in a row much more easily by searching. In fact, 114, 115, 116, 117, 118, 119, 120, and 121 are all composite.

But the method here gives us what we want with no search. And just suppose we want one hundred composites in a row. A search is difficult and time-consuming, while writing

$$101! + 2$$
$$101! + 3$$
$$101! + 4$$
$$101! + 5$$
$$101! + 6$$
$$\vdots$$
$$101! + 99$$
$$101! + 100$$
$$101! + 101$$

is not. Those three dots sure do save a lot of work and worry for mathematicians. Thus, there are consecutive primes separated by at least one hundred composites.

So what stops us? Nothing at all! If we want to show that there are consecutive primes separated by at least 10^{10} composites, it's easy. We merely note that the numbers

$$(10^{10} + 1)! + 2$$
$$(10^{10} + 1)! + 3$$
$$(10^{10} + 1)! + 4$$
$$\vdots$$
$$(10^{10} + 1)! + (10^{10} + 1)$$

are 10^{10} composites in a row; so the prime coming just before $(10^{10} + 1)! + 2$ and the prime coming just after $(10^{10} + 1)! + (10^{10} + 1)$ are consecutive primes.

Surely this section has a correct title.

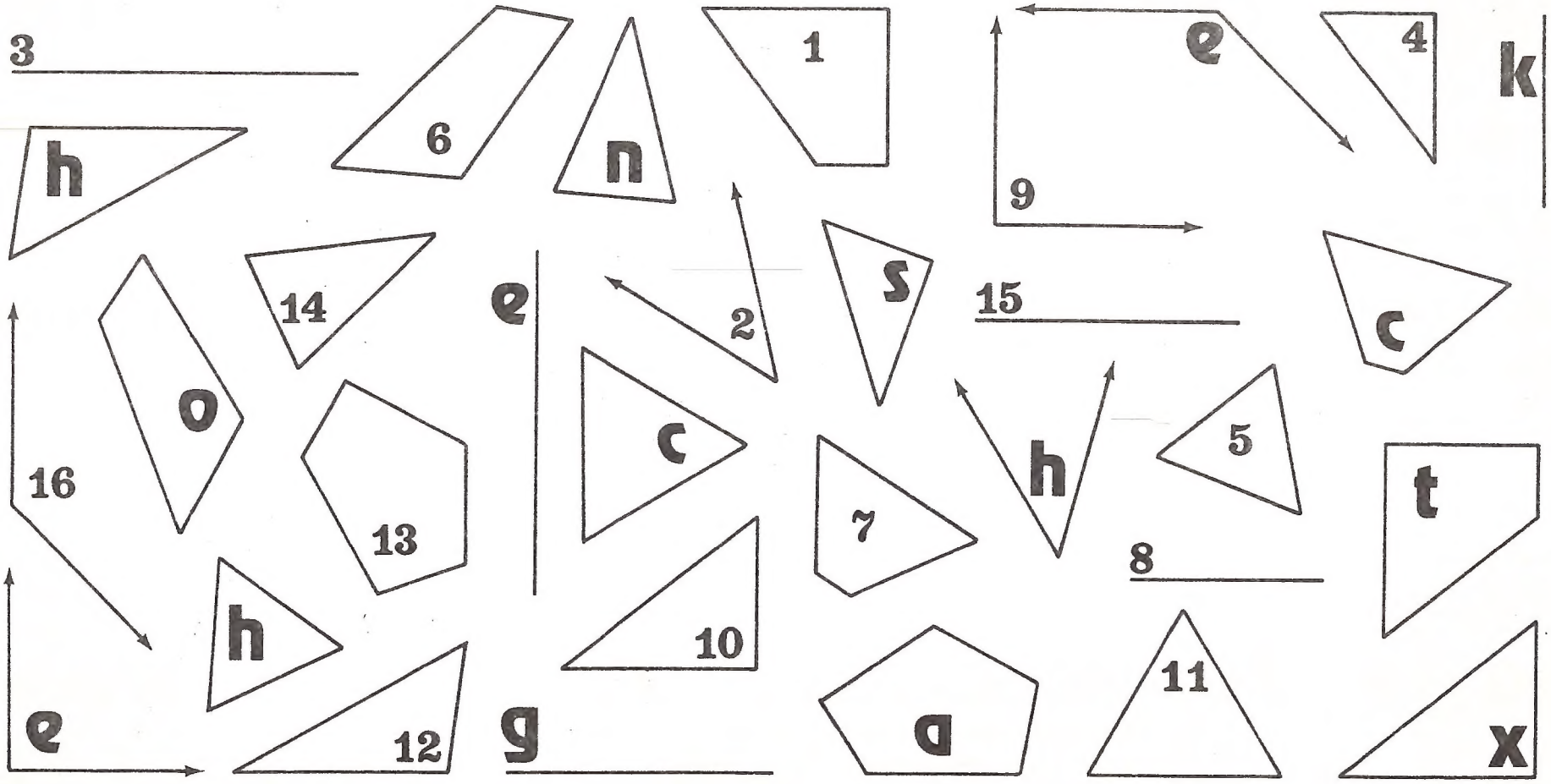
Taken from:
<u>A Panorama of Numbers</u>
Robert J. Wisner
Scott, Foresman and Company

Where Do Electric Eels Do Business?

TO ANSWER THE IMPORTANT QUESTION ABOVE, FOLLOW THESE DIRECTIONS:

Find a pair of CONGRUENT FIGURES below. One of them will have a number and the other will have a letter. The number tells you which box at the bottom of the page to put the letter in.

KEEP WORKING UNTIL YOU DISCOVER THE SHOCKING ANSWER TO THIS QUESTION. GOOD LUCK!



1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
t	h	e	s	h	o	c	k	e	x	c	h	a	n	g	e

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THE FACTOR GAME

Rules:

1. A game for two players.
2. The game uses the members shown on the board below.

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37

3. Players take alternate turns at choosing a number from the board. This number becomes a part of a player's score and is crossed from the board.
4. After one player has chosen a number, his opponent may then claim all the divisors of that number, which have not previously been used.
5. Each number on the board may be used only once, whether it is picked or claimed.
6. A player's score will be the sum of all the numbers he has picked or claimed from the board.
7. The winner is the player having the higher score at the completion of the game.

PROBLEMS

Solutions to Junior Problems will be accepted only from students in third or lower secondary years. Solutions of problems in the Open Section will be accepted from students in any grade from Matriculation downwards. Some, but not all problems in the Open Section require more advanced knowledge than those in the Junior Section.

All solutions sent in must be the result of the solvers' unaided efforts.

Where a solution is obtained by the co-operation of two participants, the solution should be jointly signed by both authors. Our published solutions sometimes have to be abbreviated, but solutions sent in should be given fully.

Problems in this issue (Vol.13, No.1). We hope to publish these solutions and the names of those who sent in good solutions in Vol.13, No.2. Solutions should be in the hands of the Editor within one month of receiving the Journal.

IMPORTANT. The solution of each problem must be written on a separate sheet, but the sheets may be sent in one envelope. Each sheet should be signed by the author(s), giving full name (in block letters), name of school and grade. You may send in solutions to one problem only or to any number of problems.

Solutions should be sent to The Editor, S.A. Mathematics Journal, Mathematics Department, University of Adelaide, Adelaide 5001.

(In some cases, Mathematical Associations outside South Australia have made alternative arrangements for the marking of solutions.)

JUNIOR PROBLEMS

1. Given a circle of radius 1 metre, and seven points lying on its circumference, show that there are two of these points which are less than 1 metre apart.

2. Show that a rectangle of given perimeter has greatest area when its sides are equal.

3. AB is the diameter of a semicircular arc and C is the midpoint of the arc. D is another point on the arc and E is its perpendicular projection on the diameter. Prove that $AC^2 = CE^2 + DE^2$.

4. Let a,b,c be positive numbers such that abc = 1. Calculate the value of

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1}$$

5. Show that in any set of 10 consecutive integers there is at least one integer which is not divisble by 2,3, 5 or 7.

6. There is a house with the interesting property that if you stand in any of its rooms you can count an even number of doors.

   Before the owner goes to bed each night, he locks all the windows, the front door, the back door and the side door. One morning he realized that his cutlery had been stolen; but when he checked the windows and doors he saw that none of them had been forced open. How did the burglar get in?

7. Let p be the smallest prime divisor of an integer n, and suppose that $p > \sqrt[3]{n}$. Show that n/p is also prime.

OPEN SECTION

01. Let f be a <u>bounded</u> real valued function defined on $[0,1] = \{x \in R;\ 0 \le x \le 1\}$ with the property:

$$f\left(\frac{x+y}{2}\right) \le \tfrac{1}{2}(f(x) + f(y))$$

for all $x,y \in [0,1]$. Suppose further that $f(0) = f(1) = 1$. Show that $f(x) \le 1$ for <u>all</u> $x \in [0,1]$.

02. Show that $\log_{10} 2$ is an irrational number.

03. The real valued function $f(x)$ is defined for all x. Furthermore, there exists a fixed number a such that for all x

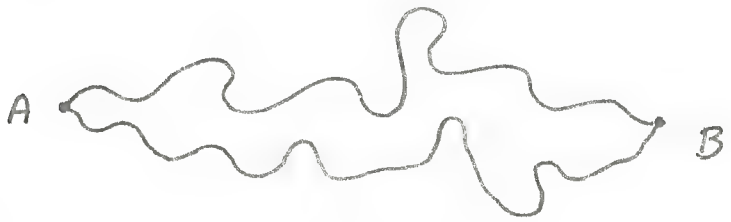
$$f(x+a) = \tfrac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

Show that $f(x)$ is periodic. (That is, there exists a real number b such that for all x

$$f(x+b) = f(x).)$$

Construct an example of such a function.

04. In a certain country there are two towns A and B, separated at some distance. There are two very narrow non-intersecting paths leading from A to B.



One day two men set out on a walk from A to B, one on each road. They held between them throughout the

journey a rope of length 2ℓ metres, and both managed to reach B without losing hold of the rope. Of course the rope may have been slack most of the time and it is not assumed that both men were necessarily travelling at the same speed. On the next day two large waggons travelled along these paths. The waggons were rather peculiar in shape. Looking from above they looked like a circle of radius ℓ metres. One waggon set out from A to reach B, the other set out from B to reach A. Show that to complete their journeys the waggons either collide or touch; in other words to complete their respective journeys one waggon will be forced off its path.

05. Show that if a is a positive integer and p is a prime number then

$$a^p \equiv (a-1)^p + 1 \pmod{p}.$$

Hence show that $a^p \equiv a \pmod{p}$ for all positive integers a.

06. Given two straight lines which are not parallel but do not meet on the sheet of paper on which they are drawn, make a construction on the paper for the bisector of the angle between them.

07. ABCDE is a convex pentagon. In how many ways is it possible to colour the edges and diagonals red or blue if no triangle determined by any three of the vertices is to have all its sides of the same colour?

FINITE GEOMETRY

P.C. Brinkworth

Everybody knows what geometry is ... or do they? A rather different point of view from the usual one may be gathered by considering finite geometries - that is, geometries having a finite number of points.

Consider the following problem. The constitution of a certain men's club lays down strict rules about forming committees:

1. For each pair of men, there is one and only one committee containing both of them.
2. Any two committees have at least one man in common.
3. There is at least one committee.
4. Every committee contains exactly three men.
5. Not all men serve on the same committee.

Problem Is it possible for the constitution to be satisfied? And if so, how many men must there be in the club?

We really have here a finite geometry in disguise - the men and their committees give us a picture or model of a geometrical system. Mathematically, we could set the system up in the following way.

Let $S = \{A,B,...\}$. Certain subsets of S (we are not saying which!) will be called m-sets. We shall assume that we know what is meant by "belongs to" and "contains".

We now lay down certain rules or axioms which must be

Mr. Brinkworth is a lecturer at Sturt College of Advanced Education.

obeyed:

1. For each pair of (distinct) elements of S , there is one and only one m-set containing both of them.
2. Any two m-sets have at least one element of S in common.
3. There exists at least one m-set.
4. Every m-set contains exactly three elements of S .
5. Not all elements of S belong to the same m-set.

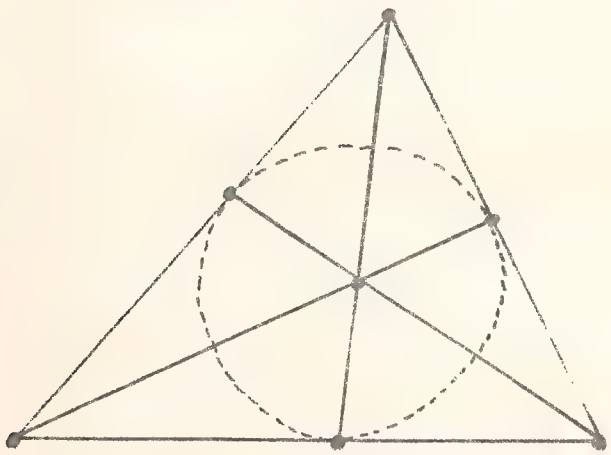
You might like to try your hand at proving the following theorems. Remember, you can use only the axioms.

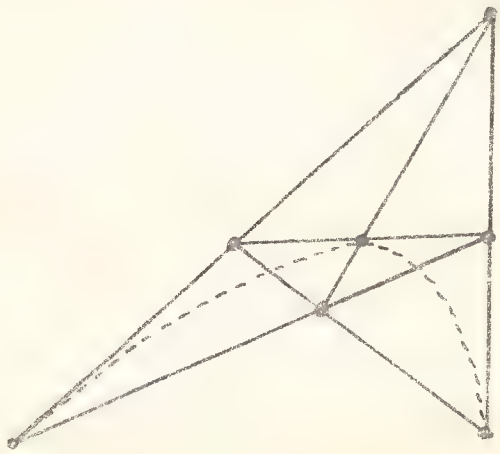
THEOREM 1. Every element of S belongs to exactly three m-sets.

THEOREM 2. S has exactly seven elements.

Now even this does not look very much like geometry. What is more, proving these theorems using logic alone, without the aid of a picture, is really quite difficult.

Suppose we write <u>point</u> in place of 'element' and <u>line</u> in place of 'm-set'. Then two geometric models which satisfy the axioms are:





The dotted line in each case represents one of the seven lines required for the geometry, even though it is not 'straight' in the ordinary sense. The theorems become more obvious when we look at a model, but we must be careful that any theorem we propose can still be deduced from the axioms.

Question What other theorems do you think might be true in this geometry?

This geometry is called Fano's seven point projective geometry.

Models don't have to be geometrical. Here is a numerical one, which may be related to the above figures by suitably numbering the points. Let the points be 0,1,2,3, 4,5,6 . Then the 'lines' are given by the columns in the following array, where line l_a contains the points a, a+1, a+3, taken modulo 7 .

l_0	l_1	l_2	l_3	l_4	l_5	l_6
0	1	2	3	4	5	6
1	2	3	4	5	6	0
3	4	5	6	0	1	2

The order of the points on each line is irrelevant.

Question Does this model satisfy the axioms? And can you label the geometrical figures in such a way that they correspond to this numerical model?

Because finite geometries are comparatively simple, they are useful for studying some of the basic questions about what geometry is. Well, then dear reader, what is geometry?

Reference

Golos, E.B., Foundations of Euclidean and Non-Euclidean Geometry (Holt, Rinehart and Winston, 1968).

PUZZLE PAGE

1. See if you can insert (single) digits in the empty squares so that the correct solution "checks out" both horizontally and vertically.

	×		−		=	11
−		×		+		−
	+		÷		=	1
+		÷		−		−
	×		−		=	6
=		=		=		=
11	−	12	+	5	=	4

2. See if you can insert signs (+,−,×,÷) in the empty squares so that the correct solution "checks out" both horizontally and vertically.

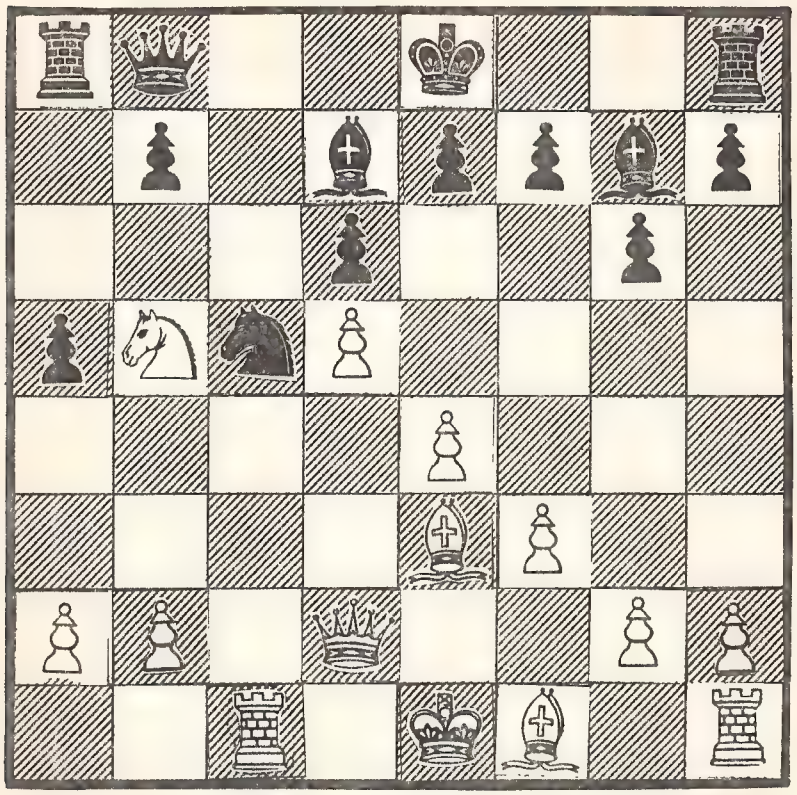
12	÷	3	+	5	=	9
+		×		÷		÷
5	×	2	−	1	=	9
−		−		−		+
8	−	3	×	2	=	2
=		=		=		=
9	−	9	+	3	=	3

CHESS COLUMN

J. van der Hoek.

One good piece of advice when playing chess is to develop one's pieces as quickly as possible and then to castle the king into safety. Failing to do this often leads to sudden loss. Here is such an example of a game between KERES (Estonia) and WESTERINEN (Finland) played in 1973. Analyse this game with your friends.

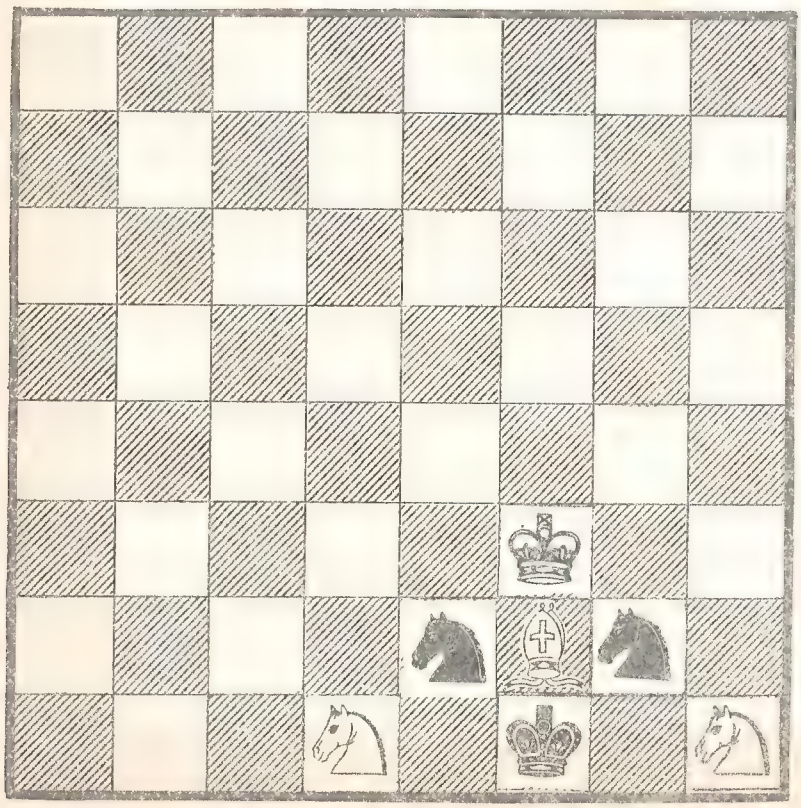
1.	P-Q4,	P-KN3	9.	R-QB1,	KN-KB3
2.	P-K4,	B-N2	10.	P-KB3,	P-QR4
3.	N-QB3,	P-Q3	11.	B-K3,	Q-B1
4.	B-KN5,	P-QB4!?(1)	12.	N-K2,	Q-N1
5.	P×P,	Q-R4	13.	N-Q4,	N-QB4
6.	Q-Q2,	Q×BP	14.	N-QN5,	N×N
7.	N-Q5!,	B-K3 (2)	15.	BP×N,	B-Q2 (4)
8.	P-QB4,	N-Q2 (3)		see diagram.	



16.	R×N!!	P×R	24.	N×B,	Q-B5 (9)
17.	P-Q6,	P×P (5)	25.	N-N5,	Q-B1 ch
18.	N×Pch,	K-B1	26.	K-B2,	Q-N2 ch (10)
19.	N×BP!	B-K3 (6)	27.	B-K2,	B-B3
20.	N-N5!	B×RP	28.	Q-Q5ch,	K-N2
21.	B×Pch,	K-N1	29.	Q×Pch,	K-R3
22.	P-QN4!	B-N6 (7)	30.	N-B7ch	Resigns.
23.	Q-Q3,	B-B2 (8)			

Notes: (1)4...N-KB3 is more usual; (2) 7...B×P?; 8. R-N1, B-N2; 9. B-K3 and 10. N-B7ch; not 9...Q-B3; 10 B-QN5!! wins (3) 8...N-KB3; 9. N×Nch etc. leaves Black with isolated QP; Black is finding it hard to castle K-side; (4) Observe now how White scatters Black's defense. You may be able to use this technique yourselves; (5) Analyse 17...B×N; (6) Why not 19...K×N; (7) Why not 22...P×P? (8) Why not 23...B-R7 or 23...P-QR5? (9) Analyse 24...K×N; (10) Why not 26...Q×N?

CHESS PROBLEM: "Not so symmetrical" White to mate Black in two moves.



SOLUTIONS TO PROBLEMS IN VOL.12, NO.2

JUNIOR SECTION

222. Insects have six legs, so we would expect that if intelligent insects develop a number-system similar to our own, the base will be 6 and not 10. Assuming that the insects also discover a "sextimal" point corresponding to our decimal point, write down how the "sextimal" expansions of $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{8}$ will appear. (The digits 7 and 8 are of course not allowable in insectland.) Under what conditions will the sextimal fractions terminate?

SOLUTION: In each division there is a zero on the left of the "sextimal" point and the remainder is

$$R = 1_{10} = 1_6$$

In each subsequent step of the division process we multiply the remainder R by

$$10_6 = 6_{10}$$

and divide the result by the given divisor. Since the remainder is less than the divisor, it is certain that the result of dividing $6R$ by the divisor, gives a quotient less than 6 hence denoting the integer part of this quotient by q, we have

$$q = q_6 = q_{10}$$

This gives the next digit in the expansion, and we obtain a new remainder R with which we continue. We come to the end of the process either when $R=0$ (i.e., when the expansion terminates) or when R assumes a value which has

occurred previously. In this case the values of q will also recur and we obtain a periodic expansion. We now tabulate the calculations as follows:

	$(R\times 10)_6$	q	R_{10}
1:5	$10_6 = 6_{10}$	1	$\underline{1}$
1:7	$10_6 = 6_{10}$	0	$6_{10} = 10_6$
	$(100)_6 = 36_{10}$	5	$\underline{1}$
1:8	$10_6 = 6_{10}$	0	$6_{10} = 10_6$
	$(100)_6 = 36_{10}$	4	$4_6 = 4_{10}$
	$(40)_6 = 24_{10}$	3	$\underline{0}$

Hence the numbers are

$$0.1111\ldots = 0.\dot{1}, \quad 0.050505\ldots = 0.\overline{05}, \quad \text{and} \quad 0.043$$

respectively. The expansion of the reduced fraction a/b (where a and b are coprime) in the sextimal system terminates if and only if b has no other prime divisor than 2 or 3, i.e.,

$$b = 2^m 3^n$$

Denoting by k the largest of m and n

$$\frac{a}{b} = \frac{A}{2^k 3^k} = \frac{A}{6^k}$$

where A is some integer. The expansion terminates after k digits to the right of the sextimal point.

D.A.C. Robertson (P.A.C. 3rd year)

223. Two cars travell-ing at uniform speed set out at noon from A and travel to B. The first car arrives at B at 3 p.m., the second at 4 p.m. At what time was the distance of the second car from B twice the distance of the first one from B?

SOLUTION: For convenience take the distance AB for unit length. In one hour the first car covers $\frac{1}{3}$ unit, and the second car covers $\frac{1}{4}$ unit. Thus after t hours the distance of the first car from B is

$$1 - t/3$$

and that of the second car is

$$1 - t/4$$

we must determine the value of t if

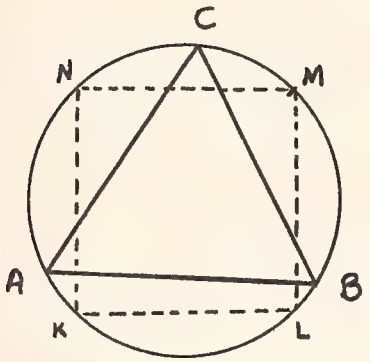
$$1 - t/4 = 2(1 - t/3)$$

Solving this equation we obtain $t = \frac{12}{5} = 2$ hrs 24 mins.

Good solution by D. Robertson.

224. An equilateral triangle and a square are inscribed in the same circle in such a way that no vertices of the triangle and square coincide. Show that amongst the seven circular arcs thus obtained there will always be one which is not longer than $\frac{1}{24}$th of the circumference of the circle.

SOLUTION:



Fix the vertices A,B,C of the equilateral triangle on the circle. The inscribed square KLMN must be placed so that its vertices are on the arcs AB, BC, CA and not at the extremities. Since 4 points must be placed on 3 arcs there will be one arc which holds 2 of the vertices of the square, say this is arc AB.

Denote by P the length of the perimeter of the circle; then

$$\text{arc } AB = \frac{P}{3} \text{ and arc } KL = \frac{P}{4} .$$

The sum of the arcs AK and BL is then

$$\frac{P}{3} - \frac{P}{4} = \frac{P}{12}$$

The smaller of the two arcs must be

$$\leq \tfrac{1}{2} \cdot \frac{P}{12} = \frac{P}{24}.$$

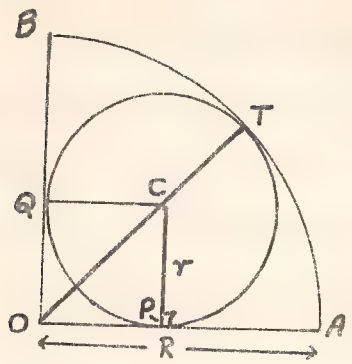
Also solved by D. Robertson.

225. A circle is to be inscribed into a quadrant of a circle of radius R (i.e. a sector with angle 90^0), so that it touches all three sides of the quadrant.

(a) Find its radius in terms of R.

(b) Show how to construct the circle using ruler and compass.

SOLUTION:



(a) Let O be the centre of the circle with radius R, C the centre of the smaller circle, r its radius and P,Q,T the points of contact of the smaller circle with the three sides A of the quadrant as shown.

Clearly OPCQ is a square of side r and by Pythagoras' theorem

$$OC = \sqrt{OP^2 + PC^2} = r\sqrt{2}$$

Also the points O,C,T are on the same straight line, hence

$$R = OT = OC + CT = r\sqrt{2} + r$$

From this

$$r = \frac{R}{\sqrt{2} + 1}$$

(b) Draw at T a tangent to the larger circle cutting OP and OQ produced at A and B respectively. Then ATB is also a tangent to the smaller circle, i.e., the latter is the inscribed circle of ΔOAB and hence its centre can be found by finding the intersection of the angular bisectors of the angles at O and A. Having found C, r = CP, i.e., the length of the perpendicular to OA.

D. Robertson (P.A.C. third year)

226. P is a point inside the acute-angled triangle ABC, and P_1, P_2, P_3 are its mirror-images in BC, CA, AB respectively. Let Q be the centre of the circumcircle of $\Delta P_1P_2P_3$, and let Q_1, Q_2, Q_3 be the mirror-images of Q in the sides of Δ ABC. Prove that P is the centre of the circumcircle of $\Delta Q_1Q_2Q_3$.

In fact, both circumcircles have the same radius.

SOLUTION:

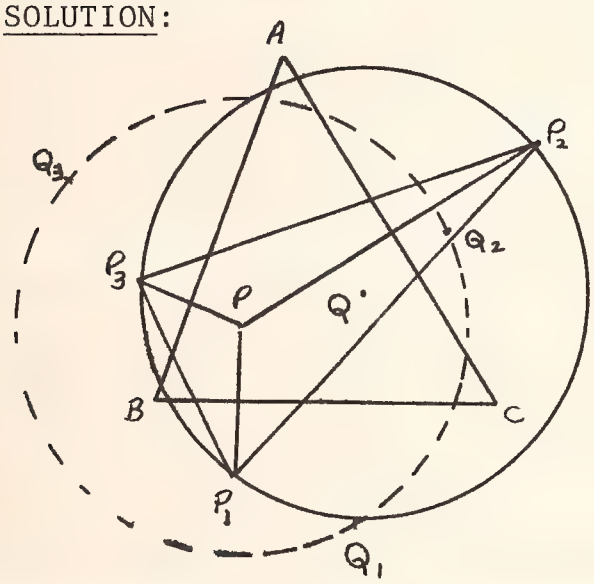


Fig. 1.

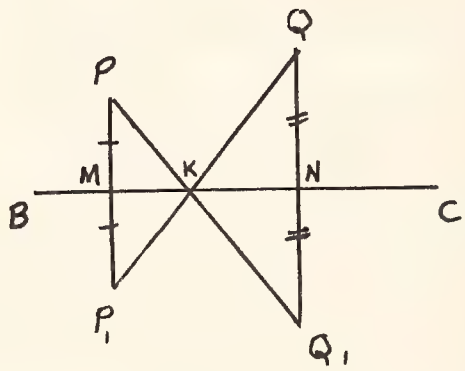


Fig. 2.

Since P_1, Q_1 are reflections of P and Q about BC. It is clear from Fig.2, that $PQ_1 = QP_1$ (More formally: Join QP_1, let its intersection with BC be K. Join K to P and Q_1, let the intersections of PP_1 and QQ_1 with BC be M and N. Then the triangles QNK and P_1MK are equiangular by construction, the triangles KMP, KMP_1 are congruent (S.A.S.), so are the triangles QNK and QNQ_1. Thus the triangles PMK and Q_1NK are also equiangular, hence P,K and Q are collinear and from the congruences $PK = P_1K$ and $KQ_1 = KQ$); we may show similarly that $PQ_2=QP_2$

and $PQ_3 = QP_3$. But $QP_1 = QP_2 = QP_3$ since Q is the centre of the circumcircle of $\Delta P_1P_2P_3$. Thus $PQ_1 = PQ_2 = PQ_3 = QP_1$, hence P is the circumcentre of the $\Delta Q_1Q_2Q_3$ and the radii of the circumcircles of Δ's $P_1P_2P_3$ and $Q_1Q_2Q_3$ are equal.

OPEN SECTION

0222. Prove that

$$\frac{2^n + (-1)^{n-1}}{3}$$

is an odd integer for integral $n \geq 1$.

SOLUTION:

$$\frac{2^1 + (-1)^0}{3} = \frac{3}{3} = 1 \text{ which is an odd integer}$$

Let $\frac{2^k + (-1)^{k-1}}{3}$ be an odd integer.

Then $\frac{2^{k+1} + (-1)^k}{3} + \frac{2^k + (-1)^{k-1}}{3} = \frac{2^k(2+1)+(-1)^{k-1}(-1+1)}{3}$

$$= 2^k$$

which is an even integer, and hence $\frac{2^{k+1} + (-1)^k}{3}$ is an odd integer.

Therefore by the principle of mathematical induction it follows that $\frac{2^n + (-1)^{n-1}}{3}$ is an odd integer for all integral $n \geq 1$.

Stephen Roberts, Marion High, Matriculation.

Good solutions also from Bevan Jarvis, Cosimo Pischettola, Mark Yudkin.

Partial solution from Peter Marshall.

0223. Find all values of m for which the quartic equation

$$x^4 - (3m+2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

SOLUTION: Let roots be $\mp a$, $\mp 3a$, where a is real

$$\Rightarrow (x+a)(x-a)(x+3a)(x-3a) = (x^2-a^2)(x^2-9a^2)$$

$$= x^4 - 10a^2x^2 + 9a^4$$

Equating coefficients, $10a^2 = 3m + 2$

$$9a^4 = m^2$$

$$\Rightarrow a^4 = \left(\frac{3m+2}{10}\right)^2 = \frac{m^2}{9}$$

$$\Rightarrow 19m^2 - 108m - 36 = 0$$

$$\Rightarrow (m-6)(19m-6) = 0$$

$$\Rightarrow m = 6 \text{ or } -6/19.$$

Partially solved by Stephen Roberts, Cosimo Pischettola.

0224. In how many ways can the word "mathematics" be spelt starting from the top and working down through the following array.

```
      M
     A A
    T T T
   H H H H
  E E E E E
 M M M M M M
  A A A A A
   T T T T
    I I I
     C C
      S
```

SOLUTION: If we can select any letter in each row, the number of ways is

$$1.2.3.4.5.6.5.4.3.2.1 = 86,400.$$

However, if we can only select a neighbouring letter directly underneath, then the number of routes to each letter is given by

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1
			21		35		35		21			
				56		70		56				
					126		126					
						252						

and hence the number of ways of spelling the word "mathematics" is 252.

Mark Yudkin, Marion High, Matriculation.

Also solved by Peter Marshall, Bevan Jarvis, D. Robertson.

0225. A wardrobe closet contains n pairs of shoes. If $2r$ shoes are chosen at random (with $2r < n$), what is the probability that there will be no complete pair?

<u>SOLUTION</u>: The probability of not picking a pair is

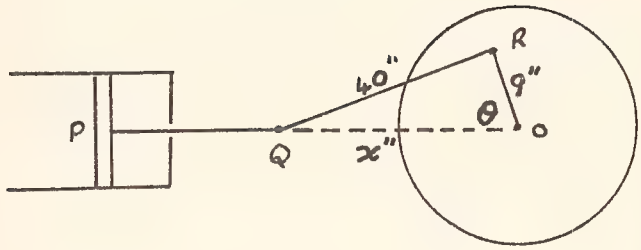
$$\frac{2^{2r}N.N-1.N-2...N-2r+1}{2N.2N-1.2N-2...2N-2r+1} = \frac{2^{2r}N!(2N-2r)!}{(N-r)!\ (2N)!}$$

Stephen Roberts, Marion High, Matriculation.

Also solved by Bevan Jarvis.

0226. The crank OR of a piston engine is driven at a constant rate of three revolutions per second. If the length of OR is 9", and the length of the connecting rod QR 3'4", find the speed of the piston when the crank is perpendicular to the connecting rod.

SOLUTION:



From the cosine rule,

$$x^2 + 9^2 - 2.9.x \cos\theta = 40^2$$

$$\Rightarrow \quad 2x.\frac{dx}{dt} - 18.\frac{dx}{dt}\cos\theta + 18.x\sin\theta.\frac{d\theta}{dt} = 0$$

When $OR \perp RQ$, $x = 41$, $\sin\theta = \frac{40}{41}$, $\cos\theta = \frac{9}{41}$, $\frac{d\theta}{dt} = 6\pi$

and hence

$$\left(41 - \frac{81}{41}\right)\frac{dx}{dt} + 9.41.\frac{40}{41}.6\pi = 0$$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{9.40.6\pi}{1600/41}$$

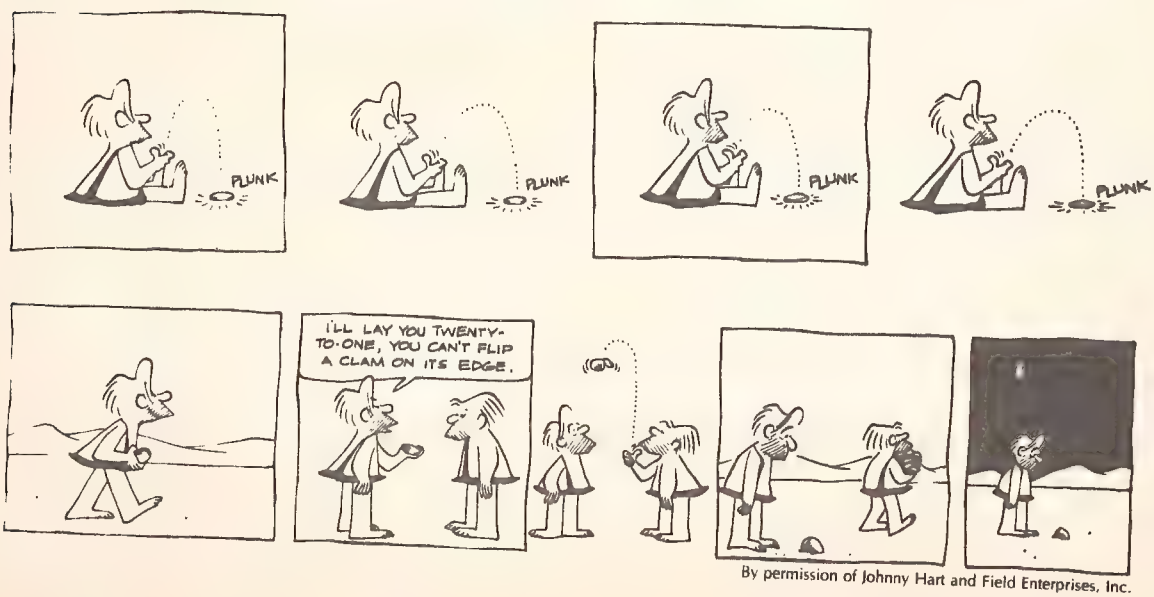
$$= -55.35\pi$$

Hence speed of piston is 55.35π inches/sec.

$\simeq$ 14.5 feet per sec.

Stephen Roberts, Marion High, Matriculation.

Also solved by Bevan Jarvis.



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NUMBER MAGIC

THIS WORKS BECAUSE:

$$7 \times 11 \times 13 = 1001$$

```
   A B C
  1 0 0 1
  -------
   A B C
  0 0 0
 0 0 0
A B C
-------
A B C,A B C
```

REPRINTED FROM AFTERMATH 3

THE THEOREM OF LIETHAGORAS

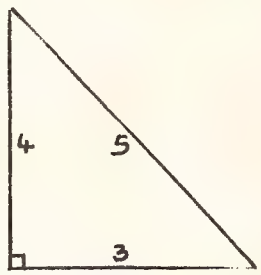
Allan White

Once upon a time Liethagoras, jealous of Pythagoras' fame, proposed a theorem. He claimed to be able to find all right-angled triangles faster than Pythagoras. His theorem was:

> "In any right-angled triangle, the square of the smallest side is equal to the sum of the other two sides."

For example:

3,4,5	$3^2 = 4 + 5$
5,12,13	$5^2 = 12 + 13$
7,24,25	$7^2 = 24 + 25$
9,40,41	$9^2 = 40 + 41$



However, the mathematicians only laughed at Liethagoras, and Pythagoras' theorem was saved.

Questions

1. Why did the mathematicians laugh? Can you find a triad that obeys Pythagoras' theorem but does not satisfy Liethagoras' theorem?
2. Find a triad that obeys Liethagoras' theorem but does not occur as the side lengths of a right-angled triangle.
3. Since Liethagoras' theorem does work for certain right-angled triangles, under what conditions is it true?

Allan White is a student at South Sydney Boys' High School.

SOLUTIONS TO PROBLEMS IN VOL. 12, NO. 3.

JUNIOR SECTION

227. Find five consecutive numbers the sum of which is a perfect square less than 100. Can you find 4 consecutive numbers the sum of which is a perfect square?

SOLUTION: (a) If the five consecutive numbers are denoted by $x-2$, $x-1$, x, $x+1$, $x+2$, the sum will be $5x$; thus the perfect square must also be a multiple of 5. The only perfect square below 100 which is also a multiple of 5 is 25, therefore $x=5$ and the numbers are 3,4,5,6,7.

(b) If the smallest of four consecutive numbers is $x-1$, then the other three are $x, x+1, x+2$, and their sum is $4x+2$. As x is an integer, $4x+2$ is even; but if $4x+2$ is a perfect square then it must be divisible by 4 which is not the case. Therefore no four consecutive numbers sum to a perfect square.

D. Robertson (P.A.C. third year).
Good solution also from H. Seppelt.

228. In how many ways can 720 be written down as a product of three positive integers different from 1?

SOLUTION: We distinguish two cases.

(a) One factor is 5, and the product of the other two is 144.

(b) One factor is a multiple of 5, greater than 5. In this case we split 144 into <u>three</u> factors different from 1,

and then multiply one of the factors by 5. For a factorisation of 144 we obtain <u>three</u> different factorisations of 720 if the factors of 144 are all different, but only two if two are equal.

e.g. $144 = 2 \times 3 \times 24$ gives

$10 \times 3 \times 24$ or $2 \times 15 \times 24$ or $2 \times 3 \times 120$

but $144 = 2 \times 2 \times 36$ gives

$10 \times 2 \times 36$ or $2 \times 2 \times 180$

Keeping this in mind, we write down the divisors of 144 in <u>order of magnitude</u>:

$$2, 3, 4, 6, 8, 9, 12, \ldots$$

(a) If 144 is split into two factors, the smallest one cannot be more than the square root of 144, i.e., it can only be one of the numbers listed above. This gives 7 possibilities, i.e.,

$$5 \times 2 \times 72, \quad 5 \times 3 \times 48 \text{ and so on.}$$

(b) If 144 is split into 3 factors, the smallest one cannot be greater than the cube root of 144, hence it cannot be greater than 4, since $6^3 > 144$.

(i) The smallest factor of 144 is 2; the remaining product is 72 which can be factorised in 5 ways with 2, 3, 4, 6, 8 being the possible smallest factors. By the introductory remarks, this gives

$$2 + (3 \times 4) = 14 \text{ possibilities}$$

for the factors of 720.

(ii) The smallest factor of 144 is 3; the remaining product 48, which can be factorised in 3 ways, since the smallest factor is at least 3 and at most 6. This gives

$$2 + (2 \times 3) = 8 \text{ possibilities.}$$

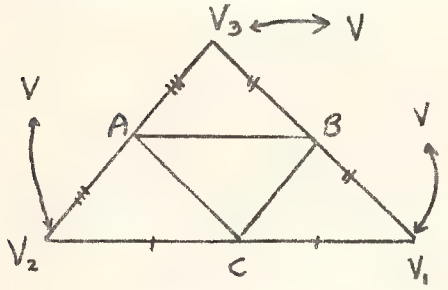
(iii) The smallest factor of 144 is 4; the remaining product is 36, this gives 2+2 possibilities since out of each of the factorisations of 144: $4 \times 4 \times 9$ and $4 \times 6 \times 6$ we can only construct two for 720.

Thus the total number of factorisations is 7+14+8+4 = 33.

Good solution by D. Robertson.

229. If the surface of a certain tetrahedron (pyramid with triangular base) is folded down in the plane, it forms a triangle. What can you say about the faces of that tetrahedron?

SOLUTION:



The line-segments obtained by folding down the same edges into the plane are obviously of equal length as marked. (The vertex V folds down into V_1, V_2, and V_3 respectively).

Since the figure obtained is a triangle, A,B,C will be the midpoints of V_2V_3, V_3V_1, and V_1V_2 respectively. Applying the midpoint theorem to the figure obtained, it follows that the four triangles shown are congruent.

Hugh Seppelt (S.P.C. 2nd year).

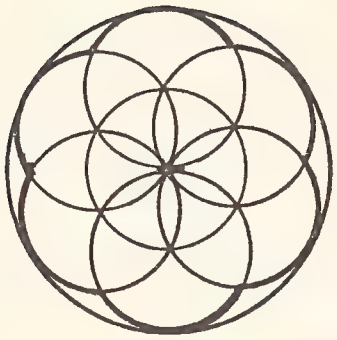
Good solution also by D. Robertson.

230. Play with your compass to reproduce the given figure. Can you tell why the inside 'flower' is formed in this way?

Which curve is longer, the one shown with thick outline or the large circle?

Prove your statement.

SOLUTION:



The two curves have the same length. Let R and r be the radii of the outside circle and the inner circles respectively. The "dark" curve is made up of 6 arcs, each of which has length $\frac{1}{3} \times 2\pi r$. So the total length of the "dark" curve is $6 \times \frac{1}{3} \times 2\pi r = 4\pi r$. But $R = 2r$, and the circumference of the outside circle is $2\pi R = 4\pi r$.

H. Seppelt and D. Robertson.

231. The hour and minute hand of the clock form a right angle. How long before they will form a right angle again?

SOLUTION: In 1 minute the minute hand travels 6^0 and the hour hand travels $\frac{1}{2}^0$, hence in each minute the angular distance between the hand increases by $6 - \frac{1}{2} = \frac{11}{2}^0$. If they are at right angles now, they will be at right angles again when the angular distance between them increases by 180^0. If this occurs after t minutes, then

$$\frac{11}{2}t = 180$$

whence $t = \frac{360}{11} = 32\frac{8}{11}$ minutes.

Good solutions by D. Robertson and H. Seppelt

OPEN SECTION

0227. Given that x, y, m, n are positive, prove that

$$x^m y^n + x^n y^m \le x^{m+n} + y^{m+n}.$$

SOLUTION:

$x \le y \Rightarrow x^m \le y^m$ and $x^n \le y^n$

$\Rightarrow x^m - y^m \le 0$ and $x^n - y^n \le 0$

$\Rightarrow (x^m - y^m)(x^n - y^n) \ge 0$

$x > y \Rightarrow x^m > y^m$ and $x^n > y^n$

$\Rightarrow x^m - y^m > 0$ and $x^n - y^n > 0$

$\Rightarrow (x^m - y^m)(x^n - y^n) > 0$

Combining, $(x^m - y^m)(x^n - y^n) \ge 0$

$\Rightarrow x^{m+n} + y^{m+n} - x^m y^n - x^n y^m \ge 0$

$\Rightarrow x^m y^n + x^n y^m \le x^{m+n} + y^{m+n}$

Vittorio Marino, Adelaide Boys' High, Matriculation.

0228. Two pairs of fair dice are rolled. What is the probability that the absolute value of the difference between the outcomes of the first pair is the same as the absolute value of the difference between the outcomes of the second pair?

SOLUTION: There are 36 possible outcomes for each pair of dice. Hence the probability for each possible value of the difference is given by

value of difference	0	1	2	3	4	5
probability	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

when two pair of fair dice are rolled, the required probability is

$$\sum_{x=0}^{5} P[\text{difference of each pair is } x]$$

$$= \left(\frac{6}{36}\right)^2 + \left(\frac{10}{36}\right)^2 + \left(\frac{8}{36}\right)^2 + \left(\frac{6}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{2}{36}\right)^2$$

$$= \frac{16}{81}.$$

David Robertson, Prince Alfred College, third year.

0229. (i) f is a function increasing on $[0,a]$ for which $f(0) = 0$. Given that f^{-1} is the inverse of f, prove that

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq a.b.$$

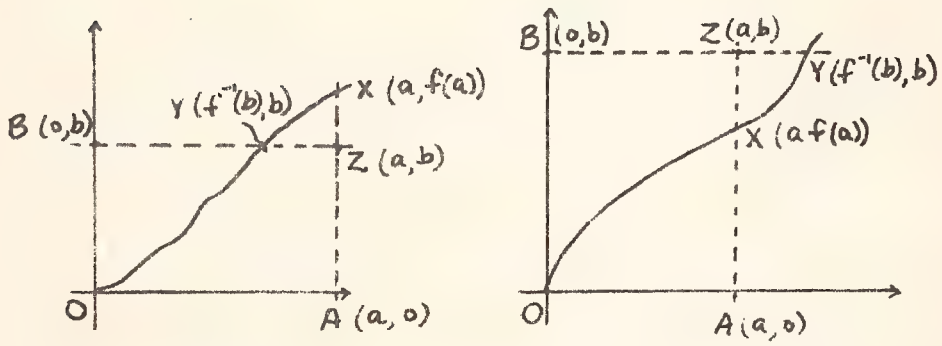
(ii) Prove that for $a > 0, b > 0, n > 1$

$$\frac{a^n}{n} + \frac{n-1}{n} b^{\left(\frac{n}{n-1}\right)} \geq ab$$

and discuss when equality occurs.

SOLUTION:

(i)



f(a)>b f(a)<b

As $\int_0^a f(x).dx$ is equal to the area of the region OAX,

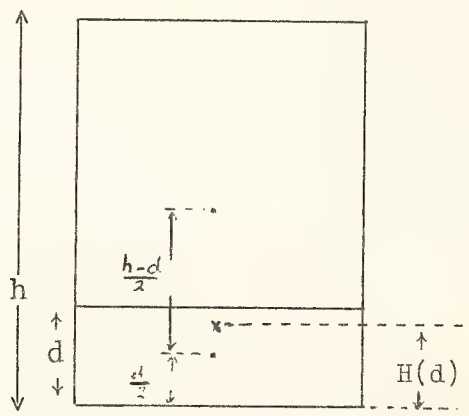
$\int_0^b f^{-1}(x)dx$ is equal to the area of the region OBY, the

required result follows. Note that equality occurs if and only if $f(a) = b$.

(ii) Let $f(x) = x^{n-1}$ in (i). Then $f^{-1}(x) = x^{\frac{1}{n-1}}$ and hence the result follows with equality if and only if $a^{n-1} = b$.

0230. Let the mass of a cylindrical soft drink can be m and that of its contents when full be c. As the drink is siphoned off, the centre of mass of the can-drink system is gradually lowered but returns to the geometric centre of the cylinder when the can is empty. How much liquid remains when the centre of mass is at its lowest point?

SOLUTION:



Let mass of can be m, that of full contents be c, and hence that of contents to a depth d is $\frac{cd}{h}$.

The centre of mass of the can-drink system is a distance

$$\frac{h-d}{2} \cdot \frac{m}{m + \frac{cd}{h}} = \frac{mh(h-d)}{2(mh+cd)}$$

above the centre of mass of the contents. Hence the height of the centre of mass above the base is

$$H(d) = \frac{d}{2} + \frac{mh(h-d)}{2(mh+cd)} = \frac{cd^2+mh^2}{2(mh+cd)}$$

$$H'(d) = \tfrac{1}{2}.\ \frac{2cd(mh+cd) - (cd^2+mh^2)c}{(mh + cd)^2}$$

$$= 0 \quad \text{when} \quad d^2 + 2h\left(\frac{m}{c}\right)d - \left(\frac{m}{c}\right)h^2 = 0$$

$$\text{i.e.} \quad d = \frac{-2h\left(\frac{m}{c}\right) \pm \sqrt{4h^2\left(\frac{m}{c}\right)^2 + 4h^2\left(\frac{m}{c}\right)}}{2}$$

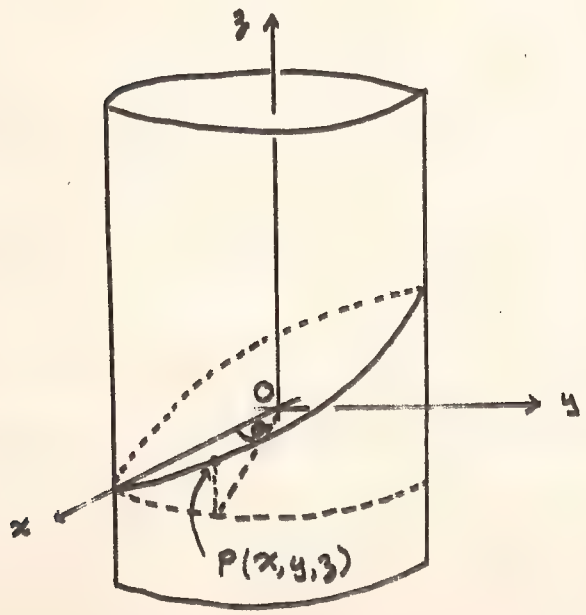
$$\text{i.e.} \quad \frac{d}{h} = -\left(\frac{m}{c}\right) \mp \sqrt{\left(\frac{m}{c}\right)^2 + \left(\frac{m}{c}\right)}$$

Hence the centre of mass is at its lowest point when

$$\frac{d}{h} = \sqrt{\left(\frac{m}{c}\right)^2 + \left(\frac{m}{c}\right)} - \left(\frac{m}{c}\right)$$

0231. The graph of $y = \sin\theta;\ 0 \le \theta \le 2\pi$ is drawn on a strip of paper which is then wrapped around a cylinder of unit radius so that the points $(0,0)$ and $(2\pi,0)$ of the graph are superimposed. Prove that the graph lies in a plane.

SOLUTION:



With respect to coordinate axes as in the diagram, the point $P(x,y,z)$ is given by

$$x = \cos\theta$$
$$y = \sin\theta$$
$$z = \sin\theta \qquad 0 \le \theta \le 2\pi$$

which lies in the plane $y-z = 0$.

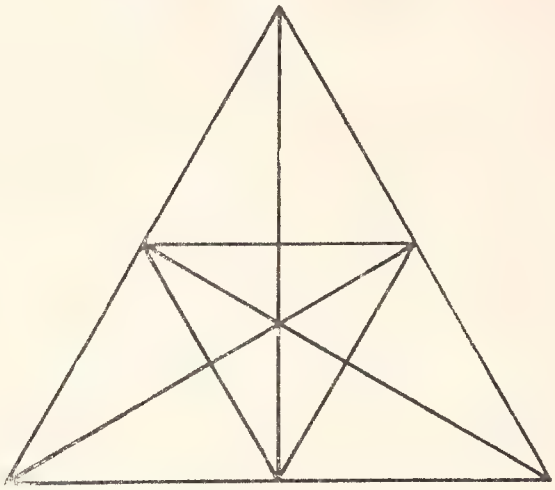
INFINITY

There is a saying that numbers do not lie, but this one does - it is a number 8 lying on its side.

NORBERT WIENER

Norbert Wiener, for years a noted professor of mathematics at the Massachussets Institute of Technology, appears to have been a model of the eccentric absent-minded professor. One day his class asked him to do a certain problem. He thought for a moment, then wrote the answer on the blackboard. The class was nonplussed, and finally a bold spirit asked, "But, Professor Wiener, is there any other way to do the problem?" The professor thought again for a moment, then brightened up and said, "Why yes, there is another way." And turning round he wrote the same answer on the board!

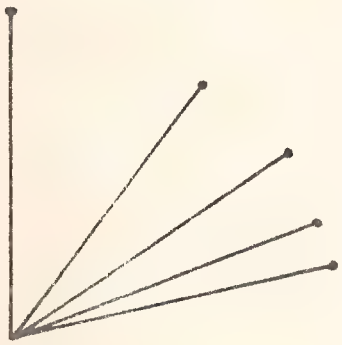
HOW MANY?



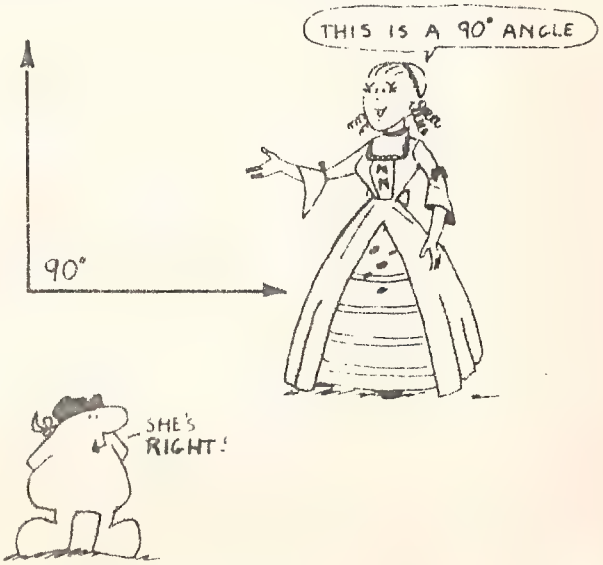
HOW MANY TRIANGLES? _____

HOW MANY PRIME NUMBERS? _____

HOW MANY SEGMENTS CONNECT ALL PAIRS OF POINTS? _____



HOW MANY DIFFERENT ANGLES LESS THAN 90? _____



BRACELETS

You may well know this sequence of numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

It is termed the Fibonacci Sequence. Can you see how one term in the sequence is obtained from the two previous terms in the sequence?

We may start with two numbers of our own choice and form our own sequence using the same method:

3, 7, 10, 17, 27, 44, 71, 115, 186, ...

1, 3, 4, 7, 11, 18, 29, 47, 76, ...

These sequences are termed Lucas Sequences.

Algebraically it is often convenient to describe a Lucas Sequence (of which the Fibonacci Sequence is a special case) as having terms

$$u_1, u_2, u_3, u_4, u_5, \ldots, u_{n-2}, u_{n-1}, u_n, u_{n+1}, \ldots$$

We need to know the values of u_1 and u_2 and then use the rule $u_n = u_{n-1} + u_{n-2}$.

For the last of the Lucas Sequences above we have

$$u_1 = 1$$
$$u_2 = 3$$
$$u_3 = u_2 + u_1 = 3 + 1 = 4$$
$$u_4 = u_3 + u_2 = 4 + 3 = 7$$
$$u_5 = u_4 + u_3 = 7 + 4 = 11.$$

Remember that to find the nth term of a Lucas Sequence we use the rule $u_n = u_{n-1} + u_{n-2}$.

Continue the following Lucas Sequences.

3, 7, 10, 17, 27, 44, 71, 115, 186 301 487 788 1275 2063

1, 9, 10, 19, 29, 48, 77, 125, ...

Look carefully at this next sequence of numbers, - it is a variation on a now familiar sequence of numbers.

1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, ...

The sequence is obtained by considering the units digits of the terms in the Fibonacci Sequence and is obtained in the following manner.

$u_1 = 1$
$u_2 = 1$
$u_3 = u_2 + u_1 = 1 + 1 = 2$
$u_4 = u_3 + u_2 = 2 + 1 = 3$
$u_5 = u_4 + u_3 = 3 + 2 = 5$
$u_6 = u_5 + u_4 = 5 + 3 = 8$
$u_7 = u_6 + u_5 = 8 + 5 = 13$ and $13 - 10 = 3$
$u_8 = u_7 + u_6 = 3 + 8 = 11$ and $11 - 10 = 1$
$u_9 = u_8 + u_7 = 1 + 3 = 4$

Whenever the sum $u_{n-1} + u_{n-2}$ exceeds 9, the value is reduced by 10.

This procedure is referred to as <u>addition modulo</u> 10.

Let us apply this new procedure to a different pair of terms.

$u_1 = 4$
$u_2 = 7$
$u_3 = u_2 + u_1 = 7 + 4 = 11$ and $11 - 10 = 1$
$u_4 = u_3 + u_2 = 1 + 7 = 8$
$u_5 = u_4 + u_3 = 8 + 1 = 9$
$u_6 = u_5 + u_4 = 9 + 8 = 17$ and $17 - 10 = 7$

Continue this sequence of numbers.

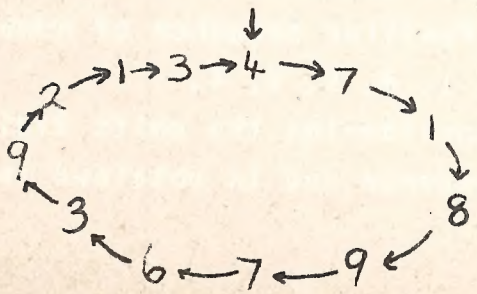
4, 7, 1, 8, 9, 7, _, _, _, _, _, _, _, _, _, _, ...

You should have obtained the sequence

4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, <u>4, 7, 1, 8, ...</u>

But notice that the last four terms listed are a repetition of the first four terms. Can you see how the rest of the sequence will continue?

We can for convenience write this sequence in the following manner.



<u>in bracelet form</u>.

We entered this bracelet at 4. But what if we had entered the bracelet at the 7 followed by the 6?
i.e. consider the sequence commencing with 7 and 6 with each term formed using the operation of addition modulo 10.

7, 6, 3, 9, 2, _, _, _, _, _, _, _, _, _, _, _, ...

Can this sequence be written in bracelet form? Does this bracelet have any similarities to the above bracelet?

Can you see that this one bracelet is generated by commencing with any two consecutive numbers in the bracelet?

Using the rule $u_n = u_{n-1} + u_{n-2}$ with the addition modulo 10, consider the bracelets generated by the sequences

2, 6, _____
8, 4, _____

These pairs of numbers generate the same bracelet but a different bracelet to the one located above. For this same rule, $u_n = u_{n-1} + u_{n-2}$ with the addition modulo 10, there are five bracelets to be found.

The others can be found by considering the sequences
1, 1, ... 2, 2, ... and 5, 5, ...

The five bracelets contain 12, 4, 60, 20 and 3 digits respectively - a total of 99 digits.
And this is to be expected.
Each digit and its successor are the first two terms of a Lucas Sequence with addition modulo 10, and there are 99 ways of choosing these first two terms - every pair of digits except 0, 0 which is of little use to us.
Note that each pair of numbers occurs together in order only once in all bracelets.

To continue our study of bracelets we can consider modifications to the rules for locating terms.
Consider $u_n = u_{n-1} + u_{n-2}$ where + is addition modulo 7.
Let us select 1, 3, ...

$u_1 = 1$
$u_2 = 3$
$u_3 = u_2 + u_1 = 3 + 1 = 4$
$u_4 = u_3 + u_2 = 4 + 3 = 0 \quad (4 + 3 - 7 = 0)$
$u_5 = u_4 + u_3 = 0 + 4 = 4$
$u_6 = u_5 + u_4 = 4 + 0 = 4$
$u_7 = u_6 + u_5 = 4 + 4 = 1 \quad (4 + 4 - 7 = 1)$
. . . .
. . . .
. . . .

Continue, to locate a bracelet of 16 digits.

The rule, $u_n = u_{n-1} + u_{n-2}$, addition modulo 7, produces three bracelets each of 16 digits.
($3 \times 16 = 48$, the total number of pairs for commencing sequences in modulo 7 in which only the 0, 1, 2, 3, 4, 5 and 6 are permitted.)
Can you find these three bracelets?

(1) Ballad
(2) Philosophizing
(3) Portrait.
(4) Description of an object.
(5) An Abstract poem
(6) Humerous verse.
(7) Sickening.
(8) Dirty
(9) Stupid.